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On the Spectral Dimension of Causal Triangulations

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Abstract We introduce an ensemble of infinite causal triangulations, called the uniform infinite causal triangulation, and show that it is equivalent to an ensemble of infinite trees, the uniform infinite planar tree. It is proved that in both cases the Hausdorff dimension almost surely equals 2. The infinite causal triangulations are shown to be almost surely recurrent or, equivalently, their spectral dimension is almost surely less than or equal to 2. We also establish that for certain reduced versions of the infinite causal triangulations the spectral dimension equals 2 both for the ensemble average and almost surely. The triangulation ensemble we consider is equivalent to the causal dynamical triangulation model of two-dimensional quantum gravity and therefore our results apply to that model.

Keywords Random graphs · Spectral dimension · Quantum gravity

1 Introduction

The behaviour of random walks, or equivalently diffusion, on random graphs has been studied intensively in recent times. The motivation for doing so has come from many different areas of physics. For example these problems play a central role in the study of random media and have been investigated both by numerical and analytic methods [7]. In this paper

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we are concerned with the ensembles of random graphs which arise in discretized quantum gravity models (see [2] for an introduction) and we will establish some exact results for two-dimensional versions of these models which are sufficiently tractable. These are of interest in their own right but of course one might hope that they can also provide insight into the higher dimensional models.

The connection between theories of gravity and ensembles of random graphs is made through the metric tensor $g_{\mu\nu}$ which is the dynamical degree of freedom. In classical general relativity the metric satisfies Einstein's equations and for any given set of consistent initial conditions there is a unique evolution of the metric in time. Quantization using the path integral formalism then amounts, at least naively, to forming a quantum amplitude describing the evolution of the metric from $g^A_{\mu\nu}$ at $t=0$ to $g^B_{\mu\nu}$ at t with amplitude

$$\langle g_B, t | g_A, t=0\rangle = \sum_{g\in\Gamma} \exp(iS[g]/\hbar), \tag{1}$$

where Γ is the set of all possible metrics satisfying $g=g_A$ at $t=0$ and $g=g_B$ at t, and $S[g]$ is the action (the natural choice for which is the Einstein–Hilbert action which in two dimensions with fixed topology consists of the cosmological constant term alone). Note that we have made a number of assumptions here concerning the consistent definition of t. To evaluate the amplitude (1) requires a systematic way of describing the set Γ. The discretized random surface is one way of doing this [2]. For simplicity consider a two-dimensional manifold with euclidean metric (so this is not really gravity which should have a lorentzian metric) and spherical topology. Such a manifold can be triangulated with $N\ge 2$ triangles; the idea is that by taking $N\to\infty$ in an appropriate way we can recover a continuum space. The metric is associated with the triangulation by supposing that all triangles are equilateral of side a and defining the geodesic distance between any two points as La where L is the number of edges in the shortest path connecting them. Then every distinct triangulation T leads to a distinct metric and the vacuum amplitude is given by

$$Z = \sum_{T\in\mathscr{P}} \exp(-S_T) \tag{2}$$

where we have set $\hbar=1$, S_T is the discretized equivalent of the continuum action, and $\mathscr{P}$ is the set of all distinct triangulations of the sphere—or equivalently the planar random graphs with all vertices having degree 3.

Many objects of interest have been calculated in this particular model which is often known as 'two-dimensional euclidean quantum gravity'; it has a scaling limit in which $a\to 0$ and $N\to\infty$ in such a way that a non-trivial continuum model results and we refer the reader to [2] for details. However there are problems with this model as a theory of gravity some of which seem to arise as a consequence of the absence of any notion of causality in the theory. The Causal Dynamical Triangulation (CDT) model was invented [1] to build in causality from the start by imposing a well defined temporal structure. This is done by restricting $\mathscr{P}$ to random triangulations which can be consistently sliced perpendicular to one direction (the time-like direction) and in which topology change is forbidden for sub-graphs lying in the other (spacelike) direction—these graphs are fully defined in Sect. 2.2 below. The idea can be applied to space-times of two or more dimensions; unfortunately it becomes progressively more difficult with increasing dimension to obtain analytic results although much has been learned by doing numerical simulations [3, 4].

The geometry of the ensembles of graphs appearing in these gravity models can be characterized in part by universal quantities of which the most basic is the dimensionality. There are different notions of dimension. The simplest one to evaluate is usually the Hausdorff dimension d_h of a graph G, which is defined provided the volume $V_G(R)$ enclosed within a ball of a radius R takes the form

$$V_G(R) \sim R^{d_h} \tag{3}$$

at large R. The spectral dimension is defined to be d_s provided the probability $p_G(t)$ that a random walker on a graph G returns to the point of origin after a time t takes the form

$$p_G(t) \sim t^{-d_s/2} \tag{4}$$

at large time. For the fractal geometries we are interested in it is not necessarily true that all definitions of dimension agree. The spectral dimension probes different aspects of the long range properties of graphs from the Hausdorff dimension; clearly it is in some sense a measure of how easy it is for a walker to travel between different regions of the graph rather than a static measure of how large those regions are. It is important to note that the definitions (3) and (4) only make sense for infinite (connected) graphs. This is obvious for (3) while for (4) it is easy to see that $p_G(t)$ tends to a non-vanishing constant for $t \to \infty$ if G is finite.

For fixed graphs which satisfy certain uniformity conditions it is known that

$$d_h \geq d_s \geq \frac{2d_h}{1+d_h} \tag{5}$$

provided both dimensions exist, see for example [10]. Those uniformity conditions are not necessarily applicable to random graphs although this relation is satisfied in at least some examples of ensemble averages of random geometries [14]. For random graphs in general the dimensions d_h and d_s can be defined either by replacing the left hand sides of (3) and (4) by their ensemble averages or, more ambitiously, by establishing that individual graphs almost surely have a definite value of d_h or d_s. We shall focus mainly on the latter point of view in this paper. The methods we employ build on those used in earlier studies of random walk on random combs [14] and on generic random trees [13, 15]. Related results on the recurrence of random planar graphs with bounded vertex degree have been obtained in [8]. In this paper we consider graphs that do not have bounded vertex degree, although they do have other special characteristics, and so in some sense extend these results.

This paper is organized as follows. In Sect. 2 the ensembles of graphs that we consider in this paper are introduced and the relationships between them and tree ensembles constructed from Galton Watson processes explained. Section 3 discusses the Hausdorff dimension of these ensembles while in Sect. 4 it is proved that the two-dimensional causal dynamical triangulation ensemble is recurrent and therefore that its spectral dimension is bounded above by 2. In Sect. 5 we prove that the spectral dimension in the related radially reduced model is exactly 2. In the final section we discuss the significance of our results.

2 Ensembles of Random Graphs

A random graph $(\mathscr{G}, \mu)$ is a set of graphs $\mathscr{G}$ equipped with a probability measure μ. In the following we assume the graphs in $\mathscr{G}$ to have a marked vertex called the *root*. We shall discuss several measures μ or μ_X and will denote the corresponding expectation by $\langle \cdot \rangle_\mu$ or $\langle \cdot \rangle_X$. The ensembles that we consider are all related to the generic random tree $(\mathscr{T}, \mu_\infty)$ which was studied in [15] and which we first review.

2.1 The Generic Random Tree

A rooted tree T is a connected planar graph consisting of vertices v of finite degree connected by edges but containing no loops; the root r is a special marked vertex connected to only one edge and the smallest rooted tree consists of the root and one other vertex. We denote the number of edges in a tree by $|T|$. The set of all trees $\mathscr{T}$ contains the set of finite trees $\mathscr{T}_f = \bigcup_{N\in\mathbb{N}} \mathscr{T}_N$ where $\mathscr{T}_N = \{T \in \mathscr{T} : |T| = N\}$ and the set of infinite trees $\mathscr{T}_\infty$.

A Galton Watson (GW) process is defined by offspring probabilities which are a sequence of non-negative numbers $p_0 \neq 0, p_1, p_2, \ldots,$ with $p_i > 0$ for at least one $i \geq 2$. They are conveniently encoded in the generating function

$$f(x) = \sum_{n=0}^{\infty} p_n x^n \tag{6}$$

which satisfies $f(1) = 1$. We call the process *critical* if $f'(1) = 1$ and *generic* if $f(x)$ is analytic in a neighbourhood of the unit disk. Assigning the probability $p_{\sigma_v -1}$ to the event that any vertex $v \neq r$ has degree σ_v a critical GW process induces a probability distribution μ_{GW} on $\mathscr{T}_f$,

$$\mu_{\mathrm{GW}}(T) = \prod_{v\in T\backslash r} p_{\sigma_v -1}. \tag{7}$$

We will call the ensemble $(\mathscr{T}_f, \mu_{\mathrm{GW}})$ a critical Galton Watson (GW) tree.

Next we define the probability distribution μ_N on $\mathscr{T}_N$ by

$$\mu_N(T) = Z_N^{-1} \prod_{v\in T\backslash r} p_{\sigma_v -1} \tag{8}$$

where

$$Z_N = \sum_{T\in\mathscr{T}_N} \prod_{v\in T\backslash r} p_{\sigma_v -1}. \tag{9}$$

We define the *single spine trees* to be the subset $\mathscr{S}$ of the infinite trees whose members consist of a single infinite linear chain $r, s_1, s_2, \ldots,$ called the *spine*, to each vertex of which are attached a finite number of finite trees by identifying their root with that vertex. An example of a single spine tree is illustrated in Fig. 1. The following result was established in [15].

Theorem 1 *Assume that μ_N is defined as above as a probability measure on $\mathscr{T}$ where $\{p_n\}$ defines a generic and critical GW process. Then*

$$\mu_N \to \mu_\infty \quad \textit{as } N \to \infty \tag{10}$$

where μ_∞ is a probability measure on $\mathscr{T}$ concentrated on the set of single spine trees $\mathscr{S}$. The generating function for the probabilities for the number of finite branches at a vertex on the spine is $f'(x)$. Moreover, the individual branches are independently and identically distributed according to the original critical GW process.

The generic random tree associated to the given GW process is by definition $(\mathscr{S}, \mu_\infty)$.

In this context convergence of measures means that integrals of continuous bounded functions on $\mathscr{T}$ converge, where continuity refers to a metric $d_{\mathscr{T}}$ according to which two

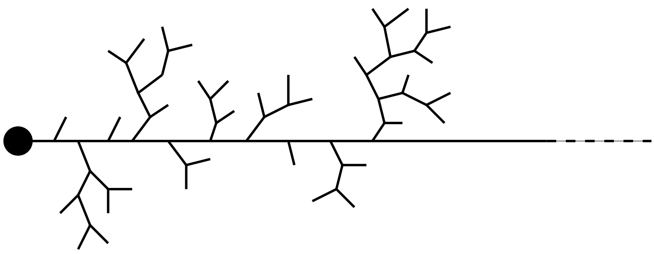


Fig. 1 Example of $T \in \mathscr{S}$

trees T and T' are close if they coincide on sufficiently large balls centred at the root. More precisely one can use

$$d_{\mathscr{T}}(T, T') = \inf\left\{\frac{1}{R} : B_R(T) = B_R(T')\right\}, \tag{11}$$

where the ball $B_R(G)$ of radius R centred at the root of a graph G is the subgraph of G spanned by the vertices at graph distance at most R from the root. The graph distance between two vertices in G is as usual the minimum number of edges in a path connecting them.

Of particular interest in the following is the so-called *uniform infinite planar tree* [12] corresponding to offspring probabilities

$$p_n = 2^{-(n+1)}, \quad n \geq 0 \tag{12}$$

which are easily seen to satisfy the requirements for a generic, critical GW tree. For this random tree we shall use the notation $\bar{\mu}_N$ for μ_N and $\bar{\mu}$ for μ_∞.

We define the height $h(v)$ of a vertex v in a graph G as the graph distance from v to the root; the height $h(\ell)$ of an edge ℓ in G as the minimum height of an end of ℓ; and the height of a finite graph G by

$$h(G) = \max_{v \in G} h(v). \tag{13}$$

Given a tree T, $D_k(T)$ is the set of vertices at height k (so that $D_0 = r$ and D_1 consists of the unique vertex which is the neighbour of the root); the number of vertices in $D_k(T)$ is denoted by $|D_k(T)|$, whereas $|B_R(T)|$ denotes the number of edges in $B_R(T)$. There are a number of useful properties of μ_{GW} and μ_∞ which follow:

Lemma 1 *For large R*

$$\mu_{\mathrm{GW}}(\{T \in \mathscr{T}_f : h(T) > R\}) = \frac{2}{f''(1)R} + O(R^{-2}). \tag{14}$$

Proof This is well known and the proof is given in e.g. [17]. □

Lemma 2 *There exists a constant $c > 0$ such that*

$$\left\langle |D_k|^{-1} \right\rangle_\infty \leq \frac{c}{k}. \tag{15}$$

Furthermore,

$$\langle |D_k|\rangle_\infty = (k-1)f''(1)+1, \quad k\geq 1, \tag{16}$$

$$\langle |B_k|\rangle_{\mathrm{GW}} = k, \quad k\geq 1, \tag{17}$$

$$\langle |B_k|\rangle_\infty = \frac{1}{2}k(k-1)f''(1)+k, \quad k\geq 1. \tag{18}$$

Proof The proof is given in [15], Proof of Lemma 5 and Appendix 2. □

2.2 Causal Triangulations

In this sub-section we define the notion of a causal triangulation (CT) and recall the definition of the model of causal dynamical triangulations (CDT) introduced in [1].

We say that a graph on n vertices $v_1, v_2, \ldots, v_n$ is a cycle if the edge set is

$$\{(v_1,v_2),(v_2,v_3),\ldots,(v_{n-1},v_n),(v_n,v_1)\}. \tag{19}$$

We include the degenerate case $n=1$ in which case we allow a loop so the unique edge is (v_1,v_1). Let G be a rooted planar triangulation (i.e. a planar graph such that all the faces, except possibly one, are triangles). Let S_0 be the root vertex of G and S_k the set of vertices at graph distance k from the root, $k=1,2,\ldots$. We say that G is a *causal triangulation* if S_k together with the edges in G which join vertices in S_k, form a cycle for $k<h(G)$ and, in the case $h(G)<\infty$, if the cycle at height $h(G)-1$ is decorated by attaching to each edge a triangle whose other two edges are not shared with any other triangle and whose vertex of order 2 belongs to the infinite face of G. The decoration of the highest cycle with triangles is not essential to the definition of CTs but it is convenient when we come to consider the measure assigned to the graphs. We denote by $\mathscr{C}$ the collection of all causal triangulations, $\mathscr{C}_K$ the elements in $\mathscr{C}$ of height K, $\mathscr{C}_f$ the collection of all triangulations in $\mathscr{C}$ of finite height and $\mathscr{C}_\infty = \mathscr{C}\setminus\mathscr{C}_f$. Note that any triangulation in $\mathscr{C}_K$ is a triangulation of the closed disk whose boundary vertices alternate in height between K and $K-1$. In this case the exterior face is not a triangle. The elements of $\mathscr{C}_\infty$ can be viewed as triangulations of the plane with the property that all the vertices at a fixed graph distance from the root form a cycle. For technical reasons that will become clear below we will assume that one of the edges emerging from the root vertex is marked and called the *root edge*. In particular, this eliminates accidental symmetries under rotations around the root vertex. An example of $G\in\mathscr{C}_4$ is shown in Fig. 2.

Given a causal triangulation G and $k<h(G)-1$ we will let Σ_k denote the subgraph of G which consists of S_k and S_{k+1} together with the edges joining them. Note that Σ_k is a triangulation of an annulus. Furthermore, we denote the number of triangles in G by $\Delta(G)$ and call it the *area* of G. Note that

$$\Delta(\Sigma_k) = |S_k| + |S_{k+1}| \tag{20}$$

(where $|S_k|$ is the number of edges in S_k and $|S_0|=0$ by definition) and that the total area of $G\in\mathscr{C}_f$ is even and equals

$$\Delta(G) = 2\sum_{k=1}^{h(G)-1} |S_k(G)|. \tag{21}$$

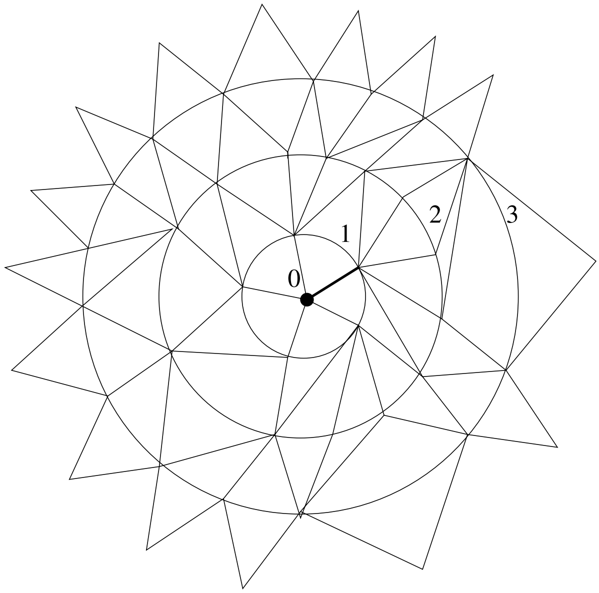


Fig. 2 Example of $G \in \mathscr{C}_4$; the numerical labels show the heights of the cycles and the root and marked edge are shown in *bold*

In the CDT model [1] each graph $G \in \mathscr{C}_f$ is assigned a weight

$$w(G) = g^{1+\Delta(G)}, \tag{22}$$

where g is the fugacity for triangles, and the grand canonical partition function is

$$Z(g) = \sum_{G \in \mathscr{C}_f} w(G). \tag{23}$$

We define the corresponding probability measure on finite causal triangulations by

$$\rho_{CT}(G) = \frac{w(G)}{Z(g)}. \tag{24}$$

The function $Z(g)$ can be computed [1] by decomposing the sum over graphs into

$$\begin{aligned} Z(g) &= \sum_{n=1}^{\infty} Z(g;n), \\ Z(g;n) &= \sum_{G \in \mathscr{C}_{n+1}} g^{1+\Delta(G)}. \end{aligned} \tag{25}$$

$Z(g;n)$ is evaluated by using (20), and counting the graphs in $\mathscr{C}_{n+1}$ by building them up successively from the slices $\{\Sigma_0, \ldots, \Sigma_{n-1}\}$. The number of ways of connecting l_{k+1} vertices in S_{k+1} with l_k vertices in S_k is $\binom{l_k+l_{k+1}-1}{l_k-1}$ and so, taking into account the marked edge,

$$Z(g;n) = g \sum_{\substack{l_i \geq 1, \\ n \geq i \geq 1}} \left(\prod_{k=1}^{n-1} \binom{l_k + l_{k+1} - 1}{l_k - 1} \right) g^{2(l_1 + \cdots + l_n)}. \tag{26}$$

Summing over l_1, and using the binomial expansion of $(1-x)^{-l}$, gives

$$Z(g;n) = g\left(\frac{X_1}{1-X_1}\right) \sum_{\substack{l_i \geq 1, \\ n \geq i \geq 2}} \left(\prod_{k=2}^{n-1} \binom{l_k + l_{k+1} - 1}{l_k - 1} \right) X_2^{l_2} g^{2(l_3 + \cdots + l_n)}, \tag{27}$$

where

$$X_{k+1} = \frac{g^2}{1-X_k}, \qquad X_1 = g^2. \tag{28}$$

Summing successively over $\{l_2, \ldots\}$, we find that

$$Z(g;n) = g \prod_{k=1}^{n} \frac{X_k}{1-X_k}. \tag{29}$$

The recursion (28) is straightforward to solve and has the following properties:

$$\begin{aligned} X_k \uparrow X^* &= \frac{1-\sqrt{1-4g^2}}{2} \quad \text{as } k \uparrow \infty \text{ for } g < \frac{1}{2}; \\ X_k &= \frac{1}{2}\frac{k}{k+1} \quad \text{at } g = \frac{1}{2}. \end{aligned} \tag{30}$$

It follows that $Z(g)$ is analytic in the disk $|g| < \frac{1}{2}$ and has a critical point at $g = \frac{1}{2}$.

To understand the nature of the critical point it is instructive to compute the average girth, defined to be the length of the cycle at half height, of finite surfaces of a fixed height

$$L(n) = Z(g;2n)^{-1} \sum_{G \in \mathscr{C}_{2n}} |S_n| g^{1+\Delta(G)}. \tag{31}$$

Slightly more involved calculations than those above yield

$$L(n) < \frac{1}{\sqrt{1-4g^2}}. \tag{32}$$

Thus for any n and $g < \frac{1}{2}$ the average surface is like a long thin tube closed off at the root end—it is essentially one-dimensional. However at $g = \frac{1}{2}$ we find that

$$L(n) = n + \frac{1}{4} + \frac{1}{4}\frac{1}{2n+1}. \tag{33}$$

which indicates that the average surface at criticality is two-dimensional. To show that the Hausdorff dimension d_h defined in (3) is indeed 2 we need to consider the tail distribution of large surfaces contributing to $Z(g)$. This only makes sense at the critical point $g = \frac{1}{2}$ since only there does the mean area of surfaces diverge due to the analyticity of $Z(g)$ for $|g| < \frac{1}{2}$. The standard way to proceed is to condition the distribution defining $Z(\frac{1}{2})$ in (23) on surfaces of fixed finite area N and take the limit $N \to \infty$ to get the appropriate ensemble of infinite surfaces. We do this in the next subsection by showing that the limit in question actually is equivalent in a precise sense to the limit obtained in Theorem 1. Subsequently, in Sect. 3, we show that $d_h = 2$ almost surely.

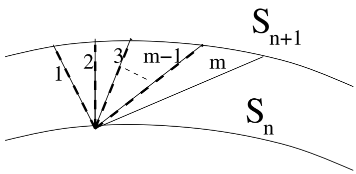


Fig. 3 The bijection from $G \in \mathscr{C}$ to $T \in \mathscr{T}$: *the dashed edges* are assigned to T

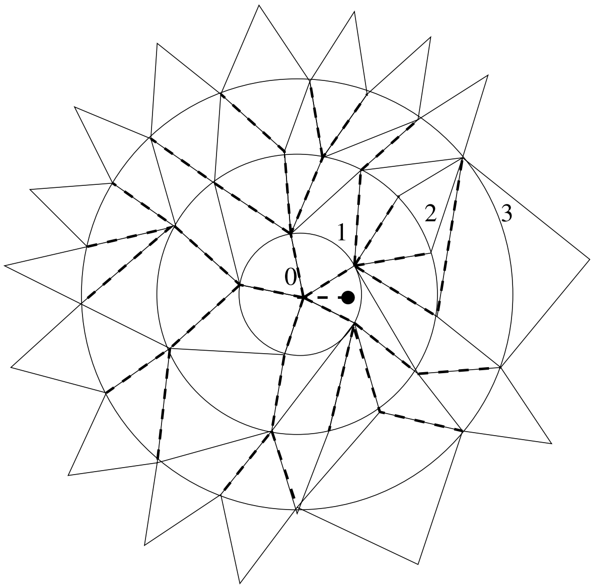


Fig. 4 The bijection from $G \in \mathscr{C}$ to $T \in \mathscr{T}$: this example shows the tree equivalent to the triangulation of Fig. 2. *The dashed lines* show the edges of the tree, including the new edge (r, S_0)

2.3 Bijection Between CT and Planar Trees

We begin by showing that causal triangulations are in one to one correspondence with rooted planar trees.

Let $G \in \mathscr{C}$. We define a planar rooted tree $T = \beta(G)$ inductively w.r.t. height of edges in the following way:

1. The vertices of T are those of G whose height is at most $h(G) - 1$ together with a new vertex r which is the root of T and whose only neighbour is S_0.
2. All edges from S_0 to $S_1(G)$ belong to T and the marked edge is the rightmost edge with respect to the edge (r, S_0).
3. For $n < h(G) - 1$ assign the edges emerging from a vertex $v \in S_n(G)$ and ending on $S_{n+1}(G)$ an ordered integer label increasing by one each time in the clockwise direction as shown in Fig. 3. All edges except the edge with highest label belong to T and have the same clockwise ordering.

Figure 4 shows an example of the application of these rules. Note that if the height of a vertex in G is n then its height in $\beta(G)$ is $n + 1$, i.e. vertices in $S_n(G)$ are in $D_{n+1}(T)$, $n < h(G) - 1$.

Conversely, let T be a rooted planar tree. Then the inverse image $G = \beta^{-1}(T)$ is obtained as follows:

1. Mark the rightmost edge connecting $D_1(T)$ and $D_2(T)$. Delete the root of T and the edge joining it to $D_1(T)$. The remaining vertices and edges of T all belong to G and $D_1(T)$ becomes S_0, the root of G.
2. For $n < h(T)$ insert edges joining vertices in $D_{n+1}(T)$ in the circular order determined by the planarity of T; this creates the sub-graphs $S_n(G)$.
3. For every vertex $v \in D_n(T)$, $2 \le n \le h(T) - 1$, that is not of order 1 in T draw an edge from v to a vertex in $S_n(G)$ such that the new edge is the most clockwise emerging from v to $S_n(G)$ and does not cross any existing edges.
4. For every vertex $v \in D_n(T)$, $2 \le n \le h(T) - 1$, of order 1 in T draw an edge from v to the unique vertex in $S_n(G)$ such that the new edge does not cross any existing edges.
5. If $h(T) < \infty$, decorate the edges of the cycle of maximum height with triangles.

A mapping equivalent to β is described in [19]. For $G \in \mathscr{C}_f$ these mappings are variants of Schaeffer's bijection [21, 23]. Indeed, deleting the edges in $S_n(G)$ for all n and identifying the vertices of maximal height $h(G)$ one obtains a quadrangulation to which Schaeffer's bijection can be applied; here the labelling of the vertices equals the height function. As we have seen, the bijection extends in this case to arbitrary infinite planar trees. For an extension to more general planar quadrangulations see [9].

This construction shows that $\beta : \mathscr{C} \to \mathscr{T}$ is a bijective map from $\tilde{\mathscr{C}}_N$, the set of causal triangulations of area $2N$, onto $\mathscr{T}_{N+1}$ and from $\mathscr{C}_\infty$ onto $\mathscr{T}_\infty$. Moreover, defining the metric $d_{\mathscr{C}}$ on $\mathscr{C}$ by

$$d_{\mathscr{C}}(G, G') = \inf\left\{\frac{1}{R+1} : B_R(G) = B_R(G')\right\}, \tag{34}$$

the map is an isometry.

Now define the finite area probability distributions ρ_N corresponding to (23) and (24) at $g = \frac{1}{2}$ by

$$\rho_N(G) = \tilde{Z}_N^{-1} 2^{-(1+\Delta(G))}, \quad G \in \tilde{\mathscr{C}}_N, \tag{35}$$

where

$$\tilde{Z}_N = \sum_{G \in \tilde{\mathscr{C}}_N} 2^{-(1+\Delta(G))}. \tag{36}$$

The following result gives the relationship between generic random trees and infinite CTs.

Theorem 2 *Let $\bar{\mu}_N$ and $\bar{\mu}$ be the measures defined by* (8) *and* (10) *corresponding to the generic, critical GW process with $p_n = 2^{-(n+1)}$, $n \ge 0$. Then*

$$\rho_N(G) = \bar{\mu}_N(\beta(G)), \quad G \in \tilde{\mathscr{C}}_N. \tag{37}$$

The limit $\rho = \lim_{N\to\infty} \rho_N$ exists and is a probability measure on $\mathscr{C}_\infty$ and is given by

$$\rho(A) = \bar{\mu}(\beta(A)) \tag{38}$$

for any event $A \subseteq \mathscr{C}_\infty$.

Proof Existence of the limit and (38) follow immediately from (37) and Theorem 1. To prove (37) consider a graph $G \in \mathscr{C}_f$ and the corresponding tree $T = \beta(G)$. Every vertex in

$S_{i+1}(G)$ has exactly one edge of T connecting it to $S_i(G)$ and therefore

$$|S_{i+1}(G)| = \sum_{v\in D_{i+1}(T)} (\sigma_v(T)-1), \quad i=0,\dots,h(G)-1. \tag{39}$$

Hence, from (21) we have

$$2^{-(1+\Delta(G))} = \frac{1}{2}\prod_{i=1}^{h(G)-1} 2^{-2|S_i(G)|} = \prod_{v\in T\setminus r} 2^{-\sigma_v}. \tag{40}$$

Comparing this with (7) identity (37) follows. □

Note that ρ_N as given by (35) is the uniform distribution on $\tilde{\mathscr{C}}_N$, that is

$$\rho_N(G) = \frac{1}{\sharp\tilde{\mathscr{C}}_N}, \quad G\in\tilde{\mathscr{C}}_N, \tag{41}$$

where $\sharp\tilde{\mathscr{C}}_N$ is the number of elements in $\tilde{\mathscr{C}}_N$ (and is given by a Catalan number). For this reason the ensemble $(\mathscr{C},\rho)$ may appropriately be called the *uniform infinite causal triangulation*. According to Theorem 1 the measure ρ is concentrated on the subset $\beta^{-1}(\mathscr{S})$ of triangulations corresponding to trees with a single spine.

A result analogous to Theorem 2 has been obtained for general planar triangulations in [5]. Finally we observe that the present relationship between trees and CTs is not the same as that introduced in [11]; in that case the trees do not in general belong to a generic random tree ensemble.

2.4 Reduced Models

We now define two simplified ensembles derived from the infinite CTs. These are useful in proving recurrence of the uniform infinite CT but also provide models which are interesting in their own right.

Let the set $\mathscr{R}$ consist of all infinite graphs constructed from the non-negative integers regarded as a graph so that n has neighbours $n\pm 1$, except for 0 which only has 1 as a neighbour, and so that there are L_n edges connecting n and $n+1$. Note that these graphs, an example of which is shown in Fig. 5, have multiple edges contrary to those considered above (they are called multi-graphs in the mathematical literature).

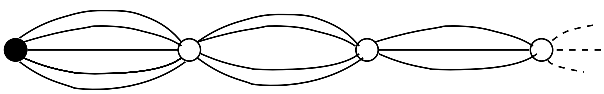
The R ensemble is defined on $\mathscr{R}$ by introducing a mapping $\gamma:\mathscr{C}_\infty\to\mathscr{R}$ which acts on $G\in\mathscr{C}_\infty$ by collapsing all the edges in $S_k, k\geq 1$, and identifying all the vertices $v\in S_k$ so there is only one vertex at each height but all the edges connecting S_k and S_{k+1} are retained. The measure on $\mathscr{R}$ is then inherited from that on $\mathscr{C}_\infty$ so that for integers $0\leq k_1<\cdots<k_m$ and positive integers $M_1,\dots,M_m$

$$\begin{aligned}\chi_{\mathrm{R}}(\{G'\in\mathscr{R}: L_{k_i}=M_i,\ i=1,\dots,m\})\\ =\rho(\{G\in\mathscr{C}_\infty: |S_{k_i}(G)|+|S_{k_i+1}(G)|=M_i,\ i=1,\dots,m\})\\ =\bar{\mu}(\{T\in\mathscr{S}: |D_{k_i+1}(T)|+|D_{k_i+2}(T)|=M_i,\ i=1\dots m\}).\end{aligned} \tag{42}$$

A related ensemble R′ is obtained by defining γ to retain only the edges connecting S_k and S_{k+1} that belong to the tree $\beta(G)$ in which case the measure on $\mathscr{R}$ is determined by

$$\chi_{\mathrm{R}'}(\{G\in\mathscr{R}: L_{k_i}=M_i,\ i=1,\dots,m\}) = \bar{\mu}(\{T\in\mathscr{S}: |D_{k_i+2}|=M_i,\ i=1,\dots,m\}). \tag{43}$$

Fig. 5 An example of $G \in \mathscr{R}$



3 Hausdorff Dimension

As already indicated in the introduction the Hausdorff dimension of a rooted infinite graph G is defined by

$$d_h = \lim_{R\to\infty} \frac{\log |B_R(G)|}{\log R} \tag{44}$$

provided the limit exists. For the uniform infinite causal triangulation we have the following result.

Theorem 3 *The Hausdorff dimension of a causal triangulation in $\mathscr{C}_\infty$ is ρ-almost surely equal to* 2.

Noting that

$$|B_{R+1}(\beta(G))| \le |B_R(G)| \le 3|B_{R+1}(\beta(G))|, \quad G \in \mathscr{C}_\infty, \tag{45}$$

this theorem is a direct consequence of Theorem 1 and the following corresponding result for generic random trees.

Proposition 1 *For any generic random tree $(\mathscr{S}, \mu_\infty)$ the Hausdorff dimension of $T \in \mathscr{S}$ is μ_∞-almost surely equal to* 2.

Proof We actually prove a slightly stronger statement which is the following: there exist positive constants C_1 and C_2 and for μ_∞-almost all trees T a constant $R_T > 0$ such that

$$C_1(\log R)^{-2} R^2 \le |B_R(T)| \le C_2 R^2 \log R \tag{46}$$

for all $R \ge R_T$.

We begin with the lower bound. In [15], Appendix 2, it is shown that there are positive constants c_0 and λ_0 such that

$$\mu_\infty(\{T : |B_R(T)| < \lambda R^2\}) \le e^{-c_0 \lambda^{-\frac{1}{2}}} \tag{47}$$

for $R > 0$ and $0 < \lambda < \lambda_0$. Hence, for every $k > 0$ we have

$$\mu_\infty(\{T : |B_R(T)| < k(\log R)^{-2} R^2\}) \le R^{-c_0 k^{-\frac{1}{2}}} \tag{48}$$

if R is sufficiently large. Choosing $k \equiv C_1$ small enough it follows that

$$\sum_{R=1}^{\infty} \mu_\infty(\{T : |B_R(T)| < C_1(\log R)^{-2} R^2\}) < \infty. \tag{49}$$

By the Borel-Cantelli lemma we conclude that

$$\mu_\infty(\{T : |B_R(T)| < C_1(\log R)^{-2} R^2 \text{ for infinitely many } R\}) = 0 \tag{50}$$

and the lower bound follows.

In order to establish the upper bound we first prove that there exist constants $C_3, C_4 > 0$ such that

$$\mu_\infty(\{T : |B_R(T)| > \lambda R^2\}) \leq C_3 e^{-C_4\lambda} \tag{51}$$

for all λ, $R > 0$. This is a slight generalization of Lemma 2.2 in [6]. Let B_R^i denote the intersection of the ball of radius R, centred at the spine vertex s_i, with the finite GW trees attached to s_i; then

$$|B_R| \leq R + \sum_{i=1}^{R} |B_R^i| \tag{52}$$

so it suffices to show that

$$\mu_\infty(\{T : |B_R^1| + \cdots + |B_R^R| > \lambda R^2\}) \leq C_3 e^{-C_4\lambda}. \tag{53}$$

Since the $|B_R^i|$ are independent and identically distributed random variables the Chebyshev inequality gives, for any $\theta > 0$,

$$\begin{aligned}\mu_\infty(\{T : |B_R^1| + \cdots + |B_R^R| > \lambda R^2\}) &= \mu_\infty(\{T : e^{\theta(|B_R^1|+\cdots+|B_R^R|)} > e^{\theta\lambda R^2}\}) \\ &\leq e^{-\theta\lambda R^2} \left\langle \prod_{i=1}^{R} e^{\theta|B_R^i|} \right\rangle_\infty = e^{-\theta\lambda R^2} \left(\left\langle e^{\theta|B_R^1|}\right\rangle_\infty\right)^R.\end{aligned} \tag{54}$$

With the notation of [15] we have

$$\left\langle e^{\theta|B_R^i|}\right\rangle_\infty = g_R(e^\theta), \tag{55}$$

where $g_R(z) = f'(f_R(z))$ and

$$f_{K+1}(z) = zf(f_K(z)), \qquad f_1(z) = z. \tag{56}$$

Clearly (56) defines f_R inductively as an increasing analytic function on $[0, 1]$ such that $f_R(0) = 0$ and $f_R(1) = 1$. By the genericity condition, each f_R is actually defined on a slightly larger interval $[0, b_R]$ where $b_R > 1$. We will now show that we can choose

$$b_R = 1 + \frac{\beta}{(1 + \alpha(R-1))^2} \tag{57}$$

and for $z \in [1, b_R]$ we have

$$f_R(z) \leq 1 + (1 + \alpha(R-1))(z-1) \tag{58}$$

for suitable constants $\alpha > 1$ and $\beta > 0$. This will imply the bound (53).

We first choose $\rho_0 > 1$ such that $f(\rho_0) < \infty$. Since $f(1) = f'(1) = 1$ there is a constant k_1 such that

$$f(z) \leq z + k_1(z-1)^2 \tag{59}$$

for $1 \leq z \leq \rho_0$. Setting $\beta \leq \rho_0 - 1$ and

$$\alpha = (1+\beta)(1+k_1\beta) \tag{60}$$

one can easily establish (58) by elementary calculations and induction.

Choosing $c > 0$ sufficiently small we have

$$f_R(e^{cR^{-2}}) \le e^{k_2/R} \tag{61}$$

by (58), where $k_2 > 0$ is a constant. Hence,

$$g_R(e^{cR^{-2}}) \le e^{k_3/R} \tag{62}$$

for a suitable constant k_3. Now taking $\theta = cR^{-2}$ in (54) we obtain the inequality (53).

The upper bound now follows in a similar way to the lower bound: from (51) we have

$$\mu_\infty(\{T : |B_R| > kR^2 \log R\}) \le \frac{C_3}{R^{C_4 k}}. \tag{63}$$

Choosing k large enough we conclude that

$$\sum_{R=1}^{\infty} \mu_\infty(\{T : |B_R| > kR^2 \log R\}) < \infty \tag{64}$$

and the Borel-Cantelli lemma gives the upper bound for μ_∞-almost every T. □

We remark that it is a trivial consequence of this result that graphs in the R ensemble or the R′ ensemble likewise have Hausdorff dimension 2 almost surely. Moreover, defining the annealed Hausdorff dimension of a random graph $(\mathcal{G}, \mu)$ by

$$d_h^{ann} = \lim_{R\to\infty} \frac{\log \langle |B_R| \rangle_\mu}{\log R}, \tag{65}$$

we have that $d_h^{ann} = 2$ for any generic random tree as a consequence of Lemma 2. It follows that this holds for the uniform infinite CT and the R and R′ ensembles as well.

4 Recurrence of the Uniform Infinite Causal Triangulation

In this section we show that random walk on graphs in the uniform infinite CT and on graphs in the R or R′ ensembles is almost surely recurrent. We start by giving a definition of recurrency. For a rooted graph G let ω be a random walk on G of length n starting at the root at time 0 and let $\omega(t)$ denote the vertex where ω is located after t steps, $t \le n$. Simple random walk is defined in the standard manner by attributing to ω the probability

$$p_\omega = \prod_{t=0}^{n-1} \sigma_{\omega(t)}^{-1}. \tag{66}$$

The return probability is given by

$$p_G(t) = \sum_{\omega:\omega(t)=r} p_\omega, \tag{67}$$

and the first return probability $p_G^0(t)$ by a similar sum restricted to walks which do not visit the root at intermediate times, $\omega(t') \ne r$ for $0 < t' < t$. Note that $p_G(t)$ and $p_G^0(t)$ are defined for $t \le n$ and are otherwise independent of n. We say that random walk on G is recurrent if the random walk in the limit $n \to \infty$ returns to r with probability 1, that is if

$$\sum_{t=1}^{\infty} p_G^0(t) = 1, \tag{68}$$

which is easily seen to be equivalent to (see (87) below)

$$\sum_{t=1}^{\infty} p_G(t) = \infty. \tag{69}$$

If G is not recurrent it is called *transient*.

There is a useful criterion for recurrency of an infinite connected graph G expressible in terms of the effective electrical resistance between the root and infinity when G is considered as an electrical network in which each edge has resistance 1. In the case of R and R′ the resistance is straightforward to define as there is only one vertex at each height and it is simply

$$R_G(r, \infty) = \sum_{k=0}^{\infty} \frac{1}{L_k(G)}. \tag{70}$$

For $G \in \mathscr{C}_\infty$ we define $R_G(r, \partial B_K(G))$ to be the resistance between the root and the vertex v_{top} of the graph obtained from $B_K(G)$ by drawing in edges between all $v \in S_K(G)$ and a single new vertex v_{top}. We then obtain $R_G(r, \infty)$ by taking K to infinity. The crucial result for our purpose is

Theorem 4 *Random walk on an infinite connected rooted graph is transient if and only if the effective resistance from the root to infinity is finite.*

Proof The result is well known and a proof is given in e.g. [20].

For R and R′ (70) is sufficiently explicit but for the infinite uniform CT we need an extra step. Define a cutset Π in an infinite rooted graph G to be a set of edges in G such that a path from the root to infinity must include at least one member of Π. Denoting the number of edges in Π by $|\Pi|$ we then have [20, 22].

Lemma 3 (Nash-Williams) *If $\{\Pi_n\}$ is a sequence of pairwise disjoint cutsets in G then*

$$R_G(r, \infty) \ge \sum_n |\Pi_n|^{-1}. \tag{71}$$

In particular, if the right hand side is infinite, then G is recurrent.

Choosing Π_n to be those edges with one end in S_n and one end in S_{n+1} and applying the lemma gives the bound for $G \in \mathscr{C}_\infty$

$$R_G(r, \infty) \ge \frac{1}{|S_1|} + \sum_{k=1}^{\infty} \frac{1}{|S_k| + |S_{k+1}|} = \sum_{k=0}^{\infty} \frac{1}{\Delta(\Sigma_k)}. \tag{72}$$

Our proof of recurrence proceeds by establishing control over the right hand sides of (70) and (72). First we need

Lemma 4 *In the* R$'$ *ensemble the probability that the number of edges in* $G \in \mathscr{R}'$ *at height* $n-1$ *exceeds a fixed value* K *is given by*

$$\chi_{\mathrm{R}'}(\{G : |L_{n-1}| > K\}) = \frac{K+n}{n}\left(1-\frac{1}{n}\right)^K, \quad n > 1, \tag{73}$$

while for R *it is given by*

$$\chi_{\mathrm{R}}(\{G : |L_{n-1}| > K\}) = \frac{K+2n-1}{2n-1}\left(1-\frac{1}{2n}\right)^K, \quad n > 1. \tag{74}$$

In the uniform infinite CT ensemble the probability that for $G \in \mathscr{C}_\infty$ *the number of triangles in* Σ_{n-1} *(equivalently the number of edges connecting* S_{n-1} *to* S_n*) exceeds* K *is given by*

$$\rho(\{G : \Delta(\Sigma_{n-1}) > K\}) = \frac{K+2n-1}{2n-1}\left(1-\frac{1}{2n}\right)^K. \tag{75}$$

Proof These results are essentially well known. To prove (73) note that from (43)

$$\chi_{\mathrm{R}'}(\{G : |L_{n-1}| > K\}) = \bar{\mu}(\{T : |D_n| > K\}) \tag{76}$$

and the result then follows from Proposition 3.6 in [12]. (Note that (73) is the statement that $|D_n|-1$ has the negative binomial distribution NegBin$(2, 1/n)$.) Using (42), and noting that $\Delta(\Sigma_{n-1}) = |S_{n-1}| + |S_n|$, we see that (75) and (74) are equivalent and, for $n \geq 2$,

$$\chi_{\mathrm{R}}(\{G : |L_{n-1}| > K\}) = \bar{\mu}(\{T : |D_n| + |D_{n+1}| > K\}). \tag{77}$$

Then using the proof of Proposition 3.6 in [12], (28) and (30), we have

$$\begin{aligned}
&\bar{\mu}(\{T : |D_n| = l_{n-1}, |D_{n+1}| = K - l_{n-1}\}) \\
&\quad = (K-l_{n-1})2^{-K+l_{n-1}-1}\binom{K-1}{l_{n-1}-1} \\
&\qquad \times \sum_{\substack{l_i \geq 1,\\ n-2 \geq i \geq 1}}^{\infty} \left(\prod_{k=1}^{n-2}\binom{l_k+l_{k+1}-1}{l_k-1}\right) 4^{-(l_1+\cdots l_{n-1})} \\
&\quad = (K-l_{n-1})2^{-(K-l_{n-1})-1}\binom{K-1}{l_{n-1}-1}(X_{n-1})^{l_{n-1}}\prod_{k=1}^{n-2}\frac{X_k}{1-X_k} \\
&\quad = \frac{(K-l_{n-1})2^{-(K-l_{n-1})}}{n(n-1)}\binom{K-1}{l_{n-1}-1}\left(\frac{n-1}{2n}\right)^{l_{n-1}},
\end{aligned} \tag{78}$$

and therefore

$$\begin{aligned}
\bar{\mu}(\{T : |D_n| + |D_{n+1}| = K\}) &= \sum_{l=1}^{K-1}\mu_{\mathrm{T}}(\{T : |D_n| = l, |D_{n+1}| = K - l\}) \\
&= \frac{K-1}{(2n-1)^2}\left(1-\frac{1}{2n}\right)^K.
\end{aligned} \tag{79}$$

The lemma follows by summing over K. (Note that (79) is the statement that $|D_n|+|D_{n-1}|-2$ has the negative binomial distribution NegBin$(2, 1/(2n))$.)

We can now establish the main result of this section:

Theorem 5 *For a graph G in the ensembles $(\mathscr{R}, \chi_{\mathrm{R}})$, $(\mathscr{R}, \chi_{\mathrm{R}'})$ or $(\mathscr{C}_\infty, \rho)$ the effective resistance between the root and infinity $R_G(r,\infty)$ is almost surely infinite and random walk therefore almost surely recurrent.*

Proof We give the proof in detail for the CT case and proceed by showing that at large enough heights n the number of triangles in slices Σ_{n-1} almost surely does not exceed the envelope function $2an\log n$ where $a>1$ is a constant. First define the event that the number of triangles in Σ_{n-1} exceeds the envelope

$$\mathscr{A}_{a,n} = \{\Delta(\Sigma_{n-1}) > 2an\log n\}, \quad n = 1, 2, \ldots. \tag{80}$$

Then from (75) we find that

$$\rho(\mathscr{A}_{a,n}) \le (1+2a\log n)n^{-a}, \tag{81}$$

and so

$$\sum_{n=1}^{\infty} \rho(\mathscr{A}_{a,n}) < \infty. \tag{82}$$

Hence, the Borel-Cantelli lemma can be applied to conclude that $\mathscr{A}_{a,n}$ occurs for at most finitely many n with probability 1, that is for all graphs G in a set of ρ-measure 1 there exists $N_G<\infty$ such that $\Delta(\Sigma_{n-1}) \le 2an\log n$ for all $n \ge N_G$. In particular, for such G we have

$$\sum_{n=1}^{\infty} \frac{1}{\Delta(\Sigma_{n-1})} \ge \sum_{n=N_G}^{\infty} \frac{1}{2an\log n} = \infty, \tag{83}$$

which, combining Lemma 3 with (70) and (72), proves Theorem 5 for $(\mathscr{R}, \chi_{\mathrm{R}})$ and $(\mathscr{C}_\infty, \rho)$. To prove the theorem for $(\mathscr{R}, \chi_{\mathrm{R}'})$ we replace (80) by

$$\mathscr{A}_{a,n} = \{L_{n-1} > an\log n\}, \quad n = 1, 2, \ldots \tag{84}$$

and proceed as above using (73) in the next step. □

5 Spectral Dimension of the R and R′ Ensembles

We start by defining the generating functions [15]

$$Q_G(x) = 1 + \sum_{t=1}^{\infty} (1-x)^{\frac{1}{2}t} p_G(t) \tag{85}$$

and

$$P_G(x) = \sum_{t=1}^{\infty} (1-x)^{\frac{1}{2}t} p_G^0(t). \tag{86}$$

The functions $Q_G(x)$ and $P_G(x)$ are related by the identity

$$Q_G(x) = \frac{1}{1 - P_G(x)}. \tag{87}$$

In particular, it follows from (68) that random walk on G is recurrent if and only if $Q_G(x)$ diverges for $x \to 0$.

By Theorem 5 random walk is almost surely recurrent for the CT, R and R′ ensembles. Assuming Q_G has asymptotic behaviour

$$Q_G(x) \sim x^{-\alpha}, \quad \alpha \in (0, 1), \tag{88}$$

for small x then the return probability, $p_G(t)$, behaves asymptotically for large time as

$$p_G(t) \sim t^{-\frac{1}{2}d_s}, \tag{89}$$

where d_s is the spectral dimension of G and is related to α by a tauberian theorem through

$$d_s = 2 - 2\alpha. \tag{90}$$

Note that if $d_s > 2$ in (89) then $Q_G(0)$ is finite and random walk on G is not recurrent. In the borderline case $d_s = 2$ we expect logarithmic corrections to the decay (89) of $p_G(t)$ at large t and, if G is recurrent, $Q_G(x)$ to be logarithmically divergent at small x. We refer the reader to, for example, [16] Sect. VI.3 and VI.11 for details on tauberian and transfer theorems. Henceforth we shall take (90) as the definition of the spectral dimension of G where

$$\alpha = \lim_{x \to 0} \frac{\log Q_G(x)}{|\log x|}, \tag{91}$$

which we assume exists.

The annealed spectral dimension d_s^{ann} for a random graph is defined in the same way as above by replacing $Q_G(x)$ in (91) by the ensemble average.

Theorem 6 *For the ensembles* $(\mathscr{R}, \chi_{\mathrm{R}})$ *or* $(\mathscr{R}, \chi_{\mathrm{R}'})$ *we have that* $d_s^{ann} = 2$. *Moreover, if* d_s *exists almost surely then its value is* 2 *almost surely.*

We give the proof for R, that for R′ being essentially identical. To prove the theorem we need

Lemma 5 *There is a constant* $c > 0$ *such that*

$$\langle Q_G(x) \rangle_{\mathrm{R}} \le c|\log x|. \tag{92}$$

Proof Let $P_G(x; n)$ denote the generating function for first return to vertex n of a random walk on a fixed graph $G \in \mathscr{R}$ which leaves n in the direction of $+\infty$ with probability 1 and let $Q_G(x; n)$ denote the generating function for the corresponding return probabilities. The generating function satisfies the recurrence relation

$$P_G(x; n-1) = \frac{(1-x)(1-u_G(n))}{1 - u_G(n) P_G(x; n)}, \tag{93}$$

where

$$u_G(n) = \frac{L_n(G)}{L_{n-1}(G) + L_n(G)} \tag{94}$$

is the probability that when the walk is at n the next step is to $n+1$. Defining $\eta_G(x;n)$ through

$$P_G(x;n) = 1 - L_n(G)^{-1}\eta_G(x;n) \tag{95}$$

and rearranging (93) gives

$$\frac{1}{\eta_G(x;n-1)} = \frac{1}{\eta_G(x;n)} + \frac{1}{L_{n-1}(G)} - \frac{xL_{n-1}(G)}{\eta_G(x;n)\eta_G(x;n-1)}. \tag{96}$$

It follows that for $N \geq n$

$$\frac{1}{\eta_G(x;n-1)} = \frac{1}{\eta_G(x;N)} + \sum_{k=n-1}^{N-1} \frac{1}{L_k(G)} - x\sum_{k=n-1}^{N-1} \frac{L_k(G)}{\eta_G(x;k)\eta_G(x;k+1)}. \tag{97}$$

Note that since $P_G(x,k) < 1$ we have $\eta_G(x;k) > 0$ and (97) implies that for $n \leq N$

$$\frac{1}{\eta_G(x;n-1)} \leq \frac{1}{\eta_G(x;N)} + \sum_{k=n-1}^{N-1} \frac{1}{L_k(G)}. \tag{98}$$

Using (95) and (87), we then obtain

$$Q_G(x;n) \leq L_n(G)\left(\frac{Q_G(x;N)}{L_N(G)} + \sum_{k=n}^{N-1} \frac{1}{L_k(G)}\right) \leq L_n(G)\left(\frac{2}{xL_N(G)} + \sum_{k=n}^{N-1} \frac{1}{L_k(G)}\right), \tag{99}$$

where we have used the trivial bound $Q_G(x;N) \leq 2x^{-1}$. We first maximise the quantity in brackets in (99) by including only those edges inherited under γ (cf. (42)) from the infinite tree whose root is at n, as shown in Fig. 6, and which is distributed according to $\bar{\mu}$ as a consequence of Theorem 1. Having done this the prefactor $L_n(G)$ is independent of the rest of the expression and taking expectation values gives

$$\begin{aligned}\langle Q_G(x;n)\rangle_{\mathrm{R}} &\leq \langle L_n(G)\rangle_{\mathrm{R}} \left\langle \frac{2}{x|D_{N-n+1}|} + \sum_{k=1}^{N-n} \frac{1}{|D_k|}\right\rangle_{\bar{\mu}} \\ &\leq c'(n+2)\left(\frac{2}{x(N-n+1)} + \sum_{k=1}^{N-n} \frac{1}{k}\right),\end{aligned} \tag{100}$$

where we have used Lemma 2 and c' is a constant. Choosing $N = [x^{-1}]$ yields

$$\langle Q_G(x;n)\rangle_{\mathrm{R}} \leq c''(n+2)|\log x| \tag{101}$$

and Lemma 5 follows by setting $n = 0$. □

Proof of Theorem 6 Theorem 5 implies that $Q_G(x)$ diverges almost surely as $x \to 0$. Since $Q_G(x)$ is a decreasing function of x it follows that $\langle Q_G(x)\rangle_{\mathrm{R}}$ and $\langle Q_G(x)\rangle_{\mathrm{R}'}$ diverge for $x \to 0$ and hence $d_s^{ann} \leq 2$. On the other hand, Lemma 5 implies $d_s^{ann} \geq 2$.

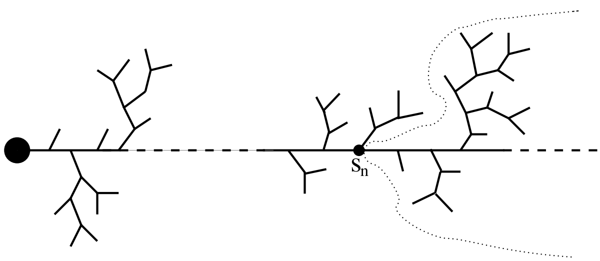


Fig. 6 Example of a tree contributing (through (42)) to (99) and (100). Only the edges to the right of the dotted boundary line are included in the sum in (100)

It remains to show that $d_s \geq 2$ almost surely. We exploit the fact that the logarithmic divergence of the ensemble average given by Lemma 5 implies that there cannot be a set of non-zero measure of graphs whose $Q_G(x)$ diverges faster than logarithmically as $x \to 0$. For $0 < x < 1$ let

$$\mathscr{A}_x = \{G \in \mathscr{R} : Q_G(x) > 1\}.$$

Since $Q_G(x)$ diverges almost surely and is decreasing in x the sets $\mathscr{A}_x$ increase to a set of measure 1 so that $\chi_{\mathrm{R}}(\mathscr{A}_x) \to 1$ for $x \to 0$. From Jensen's inequality we get

$$\begin{aligned}\int_{\mathscr{A}_x} \log Q_G(x) d\chi_{\mathrm{R}} &\leq \chi_{\mathrm{R}}(\mathscr{A}_x) \log \left((\chi_{\mathrm{R}}(\mathscr{A}_x))^{-1} \int_{\mathscr{A}_x} Q_G(x) d\chi_{\mathrm{R}} \right) \\ &\leq \chi_{\mathrm{R}}(\mathscr{A}_x) \log \left((\chi_{\mathrm{R}}(\mathscr{A}_x))^{-1} \langle Q_G(x) \rangle_{\mathrm{R}} \right). \qquad (102)\end{aligned}$$

Dividing by $|\log x|$ and using Lemma 5 then gives

$$\lim_{x \to 0} \left\langle \max \left\{ \frac{\log Q_G(x)}{|\log x|}, 0 \right\} \right\rangle_{\mathrm{R}} = 0. \qquad (103)$$

Assuming, as we do, that the limit (91) exists almost surely this shows by the dominated convergence theorem that the limit α is non-positive, that is $d_s \geq 2$ almost surely. □

Remark 1 Simple random walk on a graph $G \in \mathscr{R}$ can equivalently be considered as (non-simple) random walk on the non-negative integers with transition probabilities $\alpha_n = \frac{L_n}{L_n + L_{n-1}}$ to go from n to $n+1$ and $\beta_n = \frac{L_{n-1}}{L_n + L_{n-1}}$ to go from n to $n-1$ for $n \geq 1$ and probability $\alpha_0 = 1$ to go from 0 to 1. For general $\alpha_n, \beta_n \geq 0$ with $\alpha_n + \beta_n = 1$ such a process is called a *birth and death process* and is well known (see e.g. [18]) to be recurrent if and only if

$$\sum_{n=1}^{\infty} \frac{1}{L_n} = \infty \quad \text{where } L_n = \prod_{k=1}^{n} \frac{\alpha_k}{\beta_k}. \qquad (104)$$

Clearly, the proof of almost sure recurrence of the R and R′ ensembles could have been based on this observation instead of the Nash-Williams criterion. The estimate for the generating function $Q_G(x)$ for return probabilities obtained in the proof of Lemma 5 immediately generalises to arbitrary birth and death processes in the form

$$Q(x) \leq 1 + \sum_{n=1}^{N-1} \frac{1}{L_n} + \frac{Q(x,N)}{L_N}, \quad N \geq 1, \qquad (105)$$

where $Q(x) = Q(x;0)$ and $Q(x;N)$ denotes the generating function for return probabilities for the random walk with transition probabilities α'_n, β'_n given by $\alpha'_n = \alpha_{n+N}$. This in turn can be used to obtain an estimate on the spectral dimension of the generalised random walk in terms of the decay rate of L_n^{-1} for large n. In particular, if

$$L_n \sim n^\eta, \quad \eta < 1, \tag{106}$$

then using the bound $Q_N(x) \leq \frac{2}{x}$ and setting $N = [x^{-1}]$ one obtains

$$Q_G(x) \leq c x^{\eta - 1} \tag{107}$$

for some constant $c > 0$. This implies that the spectral dimension d_s obeys

$$d_s \geq 2\eta. \tag{108}$$

However, this bound is generally not saturated as is seen from the example of the simple random walk on the non-negative integers, where $\eta = 0$ and $d_s = 1$. But in the limiting case $\eta = 1$ we do get an optimal bound as shown above.

6 Conclusions

We have shown that the spectral dimension of the uniform infinite causal triangulation is bounded above by 2 almost surely. This result is compatible with the general result [8] that random planar graphs are almost surely recurrent if the degree of vertices is bounded and certain uniformity assumptions are satisfied. However, the uniform infinite CT does not satisfy these conditions; for example, although high degree vertices are relatively improbable, the vertex degree is not bounded. We have also shown that the Hausdorff dimension is exactly 2 almost surely so these graphs satisfy the bound

$$d_s \leq d_h \tag{109}$$

almost surely even though they do not necessarily obey the uniformity assumptions of [10]. The related R and R′ reduced models have spectral and Hausdorff dimension exactly two almost surely and therefore saturate the bound (109).

It is natural to conjecture that the spectral dimension of the uniform infinite CT equals 2 almost surely. The best lower bound known to us derives from a comparison with the uniform infinite planar tree, which is known to have spectral dimension $4/3$ [6, 15]. Indeed, deleting edges in a graph decreases the Laplace operator associated with the graph and thus decreases the spectral dimension. Hence, deleting the edges in a causal triangulation G that do not belong to the corresponding tree $\beta(G)$ we get from Theorem 2 that the spectral dimension of the uniform infinite CT is at least $4/3$. By a similar argument one can show that the spectral dimension of the R ensemble provides an upper bound on that of the uniform infinite CT. One possible strategy to prove the conjecture would be to gain better control of the error represented by this upper bound.

It is worth noting that most of the results we have proved would go through if $\beta(G)$ were in *any* generic random tree ensemble; only the proof of Lemma 4 uses the fact that we are dealing with the uniform infinite tree ensemble, but this is just a technicality. For example [11] considers an action generalized from (22) to include a dimer-like contribution, controlled by a fugacity a, which mimics some features of a higher dimensional curvature term in the action. This model at its critical point maps to an infinite random tree ensemble with the offspring probabilities

$$\begin{aligned} p_0 &= g, \\ p_n &= a^{-2} g^{n+1}, \end{aligned} \tag{110}$$

where $g = a(1+a)^{-1}$ (note that when $a = 1$ and the dimers have no effect we just recover the uniform infinite random tree); our results therefore extend the observation of universal a-independent features made in [11]. One can check that any GW tree with off-spring probabilities p_n corresponds via β to a CT model with ultralocal action in which each vertex v contributes the factor

$$p_{\sigma_f(v)-1} g^{\sigma_v} \tag{111}$$

to the weight, where $\sigma_f(v)$ is the forward degree of $v \in G$. If the GW tree is critical the CT model is critical at $g = 1$. So there is a whole universality class of surface models based on the generic random trees and all having $d_s \le 2$ and $d_h = 2$. This would be even more interesting if it were to transpire that they all have $d_s = 2$ exactly.

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